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(1) Local frames.

Set of vector fields $(X_\mu)_{\mu=1,\dots n}$ in $U \subset M$ is a frame in U
(smooth) (open)

if $(X_\mu|_p)$ is a basis in $T_p(M)$ $\forall p \in U$.

Holonomic frame $(X_\mu) \iff [X_\mu, X_\nu] = 0 \quad \forall \mu,\nu = 1,\dots n$

Holonomic frame (X_μ) in $U \iff \exists\, x^\mu$ in U s.t.

$$X_\mu = \frac{\partial}{\partial x^\mu}$$

$\frac{\partial}{\partial x^\mu}$ and $A^\mu{}_\nu = A^\mu{}_\nu(x)$ invertible matrix-valued functions in U

$\implies X_\nu = A^\mu{}_\nu \frac{\partial}{\partial x^\mu}$ is in general nonholonomic.

(2) $\Lambda^s M$ - skew-symmetric smooth tensor fields of type $\binom{0}{s}$

$$\Lambda^s M \ni \overset{s}{\omega} \iff \overset{s}{\omega}(X_1, \dots X_i, \dots X_j, \dots X_s) = -\omega(X_1, \dots X_j, \dots X_i, \dots X_s) \quad \forall i \neq j$$

Exterior differential

$$\boxed{d : \Lambda^s M \longrightarrow \Lambda^{s+1} M}$$

$$\boxed{\begin{aligned} d\overset{s}{\omega}(X_0, \dots X_s) = &\sum_{i=0}^{s} (-1)^i X_i\left(\overset{s}{\omega}(X_0, \dots \hat{X_i}, \dots, X_s)\right) + \\ &+ \sum_{0 \le i < j \le s} (-1)^{i+j} \overset{s}{\omega}([X_i, X_j], X_0, \dots \hat{X_i} \dots \hat{X_j} \dots, X_s) \end{aligned}}$$

Check that $d\omega$ is f-linear!

In particular:

$$\boxed{d\overset{1}{\omega}(X,Y) = X(\overset{1}{\omega}(Y)) - Y(\overset{1}{\omega}(X)) - \overset{1}{\omega}([X,Y])}$$

③ Wedge product:

antisymmetrization

$$Alt_s : \mathcal{X}(M)^0_s \xrightarrow{f\text{-linear}} \mathcal{X}(M)^0_s$$

$$Alt_s(\omega_1 \otimes \dots \otimes \omega_s) = \frac{1}{s!} \sum_{\sigma \in S_s} sgn(\sigma)\, \omega_{\sigma(1)} \otimes \dots \otimes \omega_{\sigma(s)}$$

in particular:

$$\Lambda^s M \ni \omega : \quad Alt_s(\omega) = \omega.$$

$$\overset{s}{\omega} \wedge \overset{t}{\omega} = \frac{(s+t)!}{s!\,t!} Alt_{s+t}(\overset{s}{\omega} \otimes \overset{t}{\omega})$$

Ex

$$dx^\mu \wedge dx^\nu = \frac{(1+1)!}{1!\,1!} Alt_{1+1}(dx^\mu \otimes dx^\nu)$$

$$= \frac{2!}{2!}(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu)$$

④ Cartan algebra $(\Lambda M, \wedge, d)$

$$\Lambda M = \bigoplus_{s=0}^{n} \Lambda^s M \quad , \quad \Lambda^0 M = \mathcal{F}(M)$$

⑤ Derivations of ΛM of degree k.

$$\mathcal{D} : \Lambda M \longrightarrow \Lambda M \quad \text{s.t.}$$

$$\mathcal{D} : \Lambda^s M \longrightarrow \Lambda^{s+k} M$$

$$\mathcal{D}(\overset{s}{\omega} \wedge \overset{p}{\omega}) = \mathcal{D}\overset{s}{\omega} \wedge \overset{p}{\omega} + (-1)^{sk} \overset{s}{\omega} \wedge \mathcal{D}\overset{p}{\omega}.$$

Example $d : \Lambda^\bullet M \longrightarrow \Lambda M$ - derivation of degree +1

Example 2

$\mathfrak{X}(M) \ni X$ defines derivation of degree -1. by

$$\begin{cases} X \lrcorner f = 0 \\ X \lrcorner df = X(f). \end{cases}$$

$$(X \lrcorner \overset{s}{\omega})(X_2, \dots X_s) = \overset{s}{\omega}(X, X_2, \dots X_s)$$

locally $\omega = \frac{1}{s!} \omega_{\mu_1 \dots \mu_s} dx^{\mu_1} \wedge \dots dx^{s}$

$X \lrcorner \omega$ from the Leibnitz rule!

Example 3

$$\underset{X}{\mathcal{L}} = X \lrcorner d + d X \lrcorner \qquad \Lambda^s M \longrightarrow \Lambda^s M$$

$$\underset{X}{\mathcal{L}}(\overset{s}{\omega} \wedge \overset{t}{\omega}) = \dots \qquad = \underset{X}{\mathcal{L}} \overset{s}{\omega} \wedge \overset{t}{\omega} + \overset{s}{\omega} \wedge \underset{X}{\mathcal{L}} \overset{t}{\omega}.$$

Example 4

Locally $\overset{2}{\omega} = \frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} \omega_{\mu\nu} (dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu)$

$$\overset{2}{\omega}(X, Y) = \frac{1}{2} \omega_{\mu\nu} (X^\mu Y^\nu - X^\nu Y^\mu)$$

$$Y \lrcorner X \lrcorner \overset{2}{\omega} = Y \lrcorner X \lrcorner \left(\frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu\right) =$$

$$= \frac{1}{2} \omega_{\mu\nu} Y \lrcorner (X^\mu dx^\nu - X^\nu dx^\mu) =$$

$$= \frac{1}{2} \omega_{\mu\nu} (X^\mu Y^\nu - X^\nu Y^\mu)$$

$$\boxed{\overset{2}{\omega}(X, Y) = (X \lrcorner \overset{2}{\omega})(Y) = Y \lrcorner X \lrcorner \overset{2}{\omega}}$$

$$(X \lrcorner \overset{2}{\omega})(Y) =$$

$$= \frac{1}{2} \omega_{\mu\nu} (X^\mu dx^\nu - X^\nu dx^\mu)(Y)$$

$$= \frac{1}{2} \omega_{\mu\nu} (X^\mu Y^\nu - X^\nu Y^\mu)$$

OK

$$\boxed{Y \lrcorner X \lrcorner \overset{2}{\omega} = \overset{2}{\omega}(X, Y)}$$

$$X_k \lrcorner \dots X_1 \lrcorner \overset{k}{\omega} = \overset{k}{\omega}(X_1, \dots X_k)$$

⑥ Maurer-Cartan theorem

X_μ - frame in $\mathcal{U}$

$[X_\mu, X_\nu] = c^\rho{}_{\mu\nu} X_\rho$ $\quad$ $c^\rho{}_{\mu\nu}$ are smooth functions in $\mathcal{U}$.

$$c^\rho{}_{\mu\nu} = -c^\rho{}_{\nu\mu}$$

$c^\rho{}_{\mu\nu} = 0$ in $\mathcal{U}$ $\iff$ $\exists\, x^\mu$ local coord. system s.t.

$$X_\mu = \frac{\partial}{\partial x^\mu}$$

ω^μ is a coframe dual to X_μ in $\mathcal{U}$ iff

$$X_\mu \lrcorner\, \omega^\nu = \omega^\nu(X_\mu) = \delta^\nu{}_\mu .$$

Thm

$$\boxed{[X_\mu, X_\nu] = c^\rho{}_{\mu\nu} \iff d\omega^\mu = -\tfrac{1}{2} c^\mu{}_{\nu\rho}\, \omega^\nu \wedge \omega^\rho}$$

Proof

$$X_\rho \lrcorner X_\nu \lrcorner d\omega^\mu = d\omega^\mu(X_\nu, X_\rho) = X_\nu(\cancel{\omega^\mu(X_\rho)}) - X_\rho(\cancel{\omega^\mu(X_\nu)}) - \omega^\mu([X_\nu, X_\rho]) =$$

$$= -c^\mu{}_{\nu\rho}$$

$$X_\rho \lrcorner X_\nu \lrcorner \left(-\tfrac{1}{2} c^\mu{}_{\alpha\beta}\, \omega^\alpha \wedge \omega^\beta\right) = \tfrac{1}{2} X_\rho \lrcorner \left(-c^\mu{}_{\nu\beta}\, \omega^\beta + c^\mu{}_{\alpha\nu}\, \omega^\alpha\right) =$$

$$= -\tfrac{1}{2}\left(c^\mu{}_{\nu\rho} + c^\mu{}_{\rho\nu}\right) = -c^\mu{}_{\nu\rho}$$

$$\boxed{d\omega^\mu = -\tfrac{1}{2} c^\mu{}_{\alpha\beta}\, \omega^\alpha \wedge \omega^\beta}$$

$\square$.

(7) Fröbenius revisited

S – distribution

$$S^* = \{ \omega \in \Lambda^1 M : \omega(X) = 0 \quad \forall X \in S \}$$

$(X_i)_{i=1,\dots m}$ frame for S $\qquad$ $i, j, k \dots = 1, \dots m$

$(\omega^\alpha)_{\alpha = m+1, \dots, n}$ frame for S^* $\qquad$ $\alpha, \beta, \gamma \dots = m+1, \dots n$

The following conditions are equivalent:

1) Through every point $p \in M$ passes precisely one integral manifold of S

$\Updownarrow$

2) $[X_i, X_j] = c^k{}_{ij} X_k$

3) $X_i = A^j{}_i (x^k, x^\gamma) \frac{\partial}{\partial x^j}$

4) $d\omega^\alpha \wedge \omega^{m+1} \wedge \dots \wedge \omega^n = 0 \quad \forall \alpha = m+1, \dots n$

5) $\omega^\alpha = B^\alpha{}_\beta (x^k, x^\gamma)\, dx^\beta$

Proof

1) $\Leftrightarrow$ 2) $\checkmark$

3) $\Rightarrow$ 5) since $X_i \lrcorner\, \omega^\alpha = 0$ and $\dim S^* = n-m$

5) $\Rightarrow$ 4) obvious

4) $\Rightarrow$ 2) $\displaystyle d\omega^\alpha = -\tfrac{1}{2} c^\alpha{}_{\beta\gamma} \omega^\beta \wedge \omega^\gamma - \tfrac{1}{2} \cancel{c^\alpha{}_{ij} \omega^i \wedge \omega^j} - \tfrac{1}{2} c^\alpha{}_{i\beta} \omega^i \wedge \omega^\beta$ $\overset{(4)}{\Downarrow}$

$$[X_i, X_j] = c^k{}_{ij} X_k + \cancel{c^\alpha{}_{ij} X_\alpha}$$

$\square$.

Rank q of a 2-form α i defined by:

$$\underbrace{\alpha\wedge\dots\wedge\alpha}_{q\text{-times}} \neq 0 \qquad \underbrace{\alpha\wedge\dots\wedge\alpha}_{(q+1)\text{-times}} = 0 \qquad 2q \leq n$$

Darboux theorem

(1) σ be a 1-form s.t. $d\sigma$ has rank $2q$.

Then there exist local coordinates

$x^1, \dots x^q, y^1, \dots y^{n-q}$ s.t.

$$\sigma = \begin{cases} x^1 dy^1 + \dots + x^q dy^q & \text{if } \sigma\wedge\underbrace{d\sigma\wedge\dots\wedge d\sigma}_{q\text{-times}} = 0 \\ x^1 dy^1 + \dots + x^q dy^q + dy^{q+1} & \text{if } \sigma\wedge\underbrace{d\sigma\wedge\dots\wedge d\sigma}_{q\text{-tims.}} \neq 0 \end{cases}$$

(2) For any 2-form α of rank q there exists a basis (ω^a) such that

$$\alpha = \omega^1\wedge\omega^2 + \omega^3\wedge\omega^4 + \dots + \omega^{2q-1}\wedge\omega^{2q}$$

If $d\alpha = 0$ then there exists coordinates $x^1, \dots x^q, y^1, \dots y^{n-q}$

$$\alpha = dx^1\wedge dy^1 + \dots + dx^q\wedge dy^q$$

Proof Sternberg S (1964)

Lectures on differential geometry (Prentice-Hall, Englewood Cliffs, NJ)